Technical Comments

Comment on "When is Hamilton's Principle an Extremum Principle?"

Cecil D. Bailey*
The Ohio State University, Columbus Ohio

IT is interesting that the authors write about Hamilton's Principle[1]; yet, no reference is made to Hamilton's papers[2,3] on the subject. Hamilton did not furnish Hamilton's Principle in the form of Eq. (1), p. 1573 of the *AIAA Journal.*[1] Hamilton furnished the "Law of Varying Action."[2-4]

$$\delta\int_{t_o}^{t_1}(T+W)\,\mathrm{d}t-(\partial T/\partial\dot q_i)\,\delta q_i\Big]_{t_o}^{t_1}=0 \qquad (1)$$

The term, $(\partial T/\partial\dot q_i)\delta q_i]_{t_o}^{t_1}$, is not in general equal to zero at t_1.[5] Hamilton pointed this out in his reference to the Principle of Least Action (which had been rigorously proven by Lagrange, but only for a stationary system), "...when this well known law of least, or as it might better be called, of stationary action, is applied to the determination of the actual motion of a system, it serves only to form, by the rules of the calculus of variations, the differential equations of motion of the second order, which can always be otherwise found. It seems, therefore, to be with reason that Lagrange, Laplace, and Poisson have spoken lightly of the utility of this principle in the present state of dynamics. A different estimate, perhaps, will be formed of that other principle which has been introduced in the present paper, under the name of the law of varying action..."[2] As demonstrated by Eqs. (4), (13), and (32) of the referenced ticle, the methods of the variational calculus still serves only to form "...the differential equations of motion of the second order, which can always be otherwise found."

The solution to the time-space path and/or configuration of a system requires more than the differential equations of motion. It requires the functional relation between the dependent space variables and the independent time and independent space variables. To further quote Hamilton,[2] "...the peculiar combination which it (i.e., the law of varying action) involves, of the principles of variations with those of partial differentials, for the determination and use of an important class of integrals, may constitute, when it shall be matured by the future labours of mathematicians, a separate branch of analysis." Thus, Hamilton predicted that which may have finally occurred.[5-9]

Substitute Eqs. (27 and 28) of the referenced article into Eq. (1) and integrate by parts to obtain

$$m_{ij}\dot q_i\delta q_j\Big]_{t_o}^{t_1}+\int_{t_o}^{t_1}(-m_{ij}\ddot q_i\delta q_j$$
$$-k_{ij}q_i\delta q_j+Q_j\delta q_j)\,\mathrm{d}t-m_{ij}\dot q_i\delta q_j\Big]_{t_o}^{t_1}=0$$

Received November 24, 1975; revision received July 25, 1975.
Index category: Structural Dynamic Analysis.

*Professor, Department of Aeronautical and Astronautical Engineering. Member AIAA.

from which, without any postulate about the initial and final conditions[5]

$$\int_{t_o}^{t_1}[-m_{ij}\ddot q_i-k_{ij}q_i+Q_j]\,\delta q_j\mathrm{d}t=0 \qquad (2)$$

Equation (2) is not merely an integral from which the differential equations of motion may be extracted from a part of the integrand. Equation (2) is the general form of the ancient principle of virtual work, and it is seen to be the expression that is an extremum and not Eq. (1) of the referenced paper. Equation (2) may contain energy dissipation terms in Q_j.[5] Q_j may be a linear or nonlinear function of the displacements, velocities, and/or time; and, it may be discontinuous.[5] Q_j in the referenced article appears to be limited to being only a function of time. Instead of yielding only the differential equations of motion, Eq. (1) or (2) permits direct solution and yields the time dependent paths and/or configurations of the system without any reference to differential equations for whatever forces that may act.[5-9]

Consider the formulation of the beam problem depicted by Eq. (12) of the referenced paper. Add to Eq. (12) of that paper time dependent applied loads and viscous damping to obtain[9]

$$\int_{t_o}^{t_1}\Big\{\int_0^L\Big[\rho A\dot w\delta\dot w-(C\dot w)\delta w$$
$$-EI(\partial^2w/\partial x^2)\,\delta(\partial^2w/\partial x^2)+Q\delta w\Big]\mathrm{d}x$$
$$+\Sigma P_i\delta w_i\Big\}\,\mathrm{d}t-\int_0^L\rho A\dot w\delta w\,\mathrm{d}x\Big]_{t_o}^{t_1}=0 \qquad (3)$$

Note that no part of the integrand in Eq. (3) constitutes the differential equation of motion. Equation (3) represents the virtual work of the forces acting on and within the system. It yields directly the time-space displacement of the beam for any specified forces and displacement constraints consistent with the assumptions made in the formulation of the beam theory used. No theory of differential equations is needed from start to finish.[8-9]

It appears to this writer that the question, "When is Hamilton's Principle an Extremum Principle?", becomes a moot point when it is seen that Hamilton did not furnish the equation that is now called Hamilton's principle, but instead, furnished the law of varying action.[2-4] From this law, it is clear that the integral which constitutes the first term of Hamilton's law is an extremum only when

$$(\partial T/\partial\dot q_i)\,\delta q_i\Big]_{t_o}^{t_1}=0$$

which, in real life is a rare occasion, indeed. It is also clear that the expression which is always an extremum is the time integral of the virtual work of the system. For the mechanics of solids, it is completely equivalent to Hamilton's law of varying action.[5]

A very elementary example will demonstrate the profound difference between Hamilton's principle, Eq. (1) of the referenced paper, and Hamilton's law, Eq. (1) of this Comment. The problem is to obtain the functional relation between the dependent space variable and the independent time variable for a particle of mass m under the action of a constant positive force F directly from Eq. (1) of the referenced paper without any use of Eq. (2) of this paper or reference to the theory of differential equations. At t_o the displacement and velocity are x_o and $\dot{x}_o$, respectively. For this problem, Hamilton's principle is

$$\delta \int_{t_o}^{t_I} (m\dot{x}^2 + 2Fx)\,\mathrm{d}t = 0 \tag{4}$$

This is probably the most elementary of all motion problems in which an acceleration is involved. Yet, the functional relation sought, $x = x_o + \dot{x}_o t + (F/2m)t^2$, is not directly available from Hamilton's principle. This exact solution is immediately obtainable from Hamilton's law, Eq. (1) of this paper, directly, without any reference to differential equations.[5-7]

References

[1] Smith, D. R. and Smith, C. V., Jr., "When is Hamilton's Principle an Extremum Principle?" *AIAA Journal,* Vol. 12, Nov. 1974, pp. 1573-76.

[2] Hamilton, W. R., "On a General Method in Dynamics," *Philosophical Transactions of the Royal Society of London,* Vol. 124, 1934, pp. 247-308.

[3] Hamilton, W. R., "Second Essay on a General Method in Dynamics," *Philosophical Transactions of the Royal Society of London,* Vol. 125, 1835, pp. 95-144.

[4] Yourgrau, W. and Mandelstam, S., *Variational Principles in Dynamics and Quantum Theory,* W. B. Saunders Co., Philadelphia, 1968, 3rd. ed.

[5] Bailey, C. D., "A New Look at Hamilton's Principle," *Foundations of Physics* Vol. 5, No. 3, Sept. 1975, pp. 433-451.

[6] Bailey, C. D., "Application of Hamilton's Law of Varying Action," *AIAA Journal*, Vol. 13, to be published.

[7] Bailey, C. D., "The Method of Ritz Applied to the Equation of Hamilton," *Computer Methods in Applied Mechanics and Engineering*, Vol. 7, No. 2, 1976, to be published.

[8] Bailey, C. D., "Exact and Direct Analytical Solutions to Vibrating Systems with Discontinuities," *Journal of Sound and Vibration*, Vol. 44, No. 1, Jan., 1976, to be published.

[9] Bailey, C. D., "Hamilton, Ritz and Elastodynamics," ASME *Journal of Applied Mechanics*, to be published.

Comment on "When is Hamilton's Principle an Extremum Principle?"

Stephen F. Felszeghy*

Hughes Aircraft Company, Culver City, Calif.

IN Ref. 1, D.R. Smith and C.V. Smith Jr. consider the question of when is Hamilton's principle an extremum principle. In developing their answer, the authors begin with the assumption that Hamilton's principle is a variational principle. It seems appropriate, therefore, to remark that Hamilton's principle cannot always be interpreted as a variational principle.

Hamilton's principle can be a variational principle if the system is holonomic. In other works, the equivalence

$$\int_{t_0}^{t_I} (\delta T - \delta V)\,dt = \delta \int_{t_0}^{t_I} (T - V)\,dt \tag{1}$$

cannot be established unless the system is holonomic. Furthermore, when Hamilton's principle is a variational principle, it states only that the actual motion renders the integral on the right-hand side of Eq. (1) stationary. These facts are discussed at length in Refs. 2 and 3.

Received February 2, 1975.

Index category: Structural Dynamic Analysis.

*Senior Staff Engineer. Member AIAA.

References

[1] Smith, D.R. and Smith Jr., C.V., "When is Hamilton's Principle an Extremum Principle?" *AIAA Journal*, Vol. 12, Nov. 1974, pp. 1573-1576.

[2] Lanczos, C., *The Variational Principles of Mechanics*, University of Toronto Press, Toronto, 1962, p. 114.

[3] Pars, L.A., *A Treatise on Analytical Dynamics*, Wiley, New York, 1965, pp. 34-37, 528-530.

Reply by Authors to S. F. Felszeghy and C. D. Bailey

Donald R. Smith*

University of California, San Diego, Calif.

and

C. V. Smith Jr.†

Georgia Institute of Technology, Atlanta, Ga.

MR. Felszeghy is not strictly correct in stating that we "begin with the assumption that Hamilton's principle is a variational principle." More precisely, we state[1] only that "For the restricted but commonly occurring problem of determining the motion of a conservative system, ..." a variational principle (in the usual meaning of that term) does exist; and the entire paper considers only this restricted problem. Of course, we are aware of the nonextremal nature of the variational principle (our Ref. 1 and, indeed, the entire paper); but this known fact has been overlooked in much of the recent literature on the subject, as noted and documented in the paper.

From our point of view, the only requirement for a variational principle is the existence of a single function V from which both external and internal forces, not including forces of constraint, can be derived. If the system is nonholonomic the virtual displacements will be required to satisfy the constraints; and we have a variational principle with subsidiary conditions (p. 85, our Ref. (4)).

The purpose of our paper is clearly stated in the paper and includes the following goals: 1) to publicize the known nonextremal nature of Hamilton's principle in variational form; 2) in the case of vibration of discrete, conservative, linearly elastic systems, to provide a direct and elementary proof of the known fact that Hamilton's principle *is* a true extremum principle over short time periods; and 3) to give a precise characterization of the maximum length of the time period over which the important result of 2) is guaranteed to be valid.

In order to achieve these goals, it was not necessary to reference Hamilton's original papers or to consider the historical question of whether what is today commonly called Hamilton's principle is indeed what Hamilton presented in 1834 and 1835. Also it was unnecessary to discuss generalizations involving forces which are nonlinear functions of displacement or viscous damping terms.

Professor Bailey states that the term $(\partial T/\partial q_i)\delta q_i |_{t_0}^{t_I}$ is not zero in real life. There would appear to be come confusion between the real displacements which a system actually experiences and the virtual, or fictitious, displacements. Real displacements satisfy equations of motion; fictitious

Received April 1, 1975.

Index category: Structural Dynamic Analysis.

*Associate Professor, Department of Mathematics.

†Associate Professor, School of Aerospace Engineering. Member AIAA.